Application of Statistical Experimental Design Techniques to Flight-Test Programs

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The increasing sophistication and complexity of modern airborne vehicles, coupled with continuing customer pressure for reduced development costs, necessitate the use of more refined test and data analysis techniques in order to remain efficient and competitive. The use of statistical experimental design techniques for achieving this increased efficiency in flight-test programs is discussed. Experimental designs can be used in flight-test programs for two purposes: 1) to reduce the number of tests required to obtain a given amount of data, thereby reducing costs; and 2) to obtain more useful data from a given number of tests in situations where the number of tests cannot be reduced appreciably. Investigation has shown that a small number of standard experimental designs is adequate to cover almost all flight testing requirements. Four general designs have been developed for use in flight testing. Computer programs to process the data from these designs have been established. One ancillary benefit of using experimental designs is the ability to conduct much of the analysis of test data on high-speed digital computers. Several typical flight-test programs are used to illustrate these techniques and the increased efficiency that can be realized through their use.

Nomenclature

b = total number of blocks
b_i = effect of the ith block
b_i' = error of the ith block
B_i = total output of the ith block
e_{ij} = experimental error in measuring the output from the jth treatment in the ith block
K = block size
P, Q, U, V, W = unknown coefficients
r = number of times the output from each factor combination or treatment is measured
t = total number of treatments or factor combinations
t_j = effect of the jth treatment
$\bar{X}$ = average value of X_{ij} over all i and j
X_{ij} = output in the ith block with the jth treatment
λ_i = number of blocks in which treatments of the ith set appear together in an incomplete block design
σ^2 = variance
μ = over-all mean

Vector quantities

E = an $N \times 1$ vector of errors
Y = an $N \times 1$ vector of observed values
β = a $K \times 1$ vector of coefficients
$\hat{\beta}$ = minimum variance, unbiased estimate of β

Introduction

In order to remain competitive, aerospace companies must use more efficient test and data analysis techniques to counteract increasing costs and complexity of flight tests. This paper discusses the use of statistical experiment design techniques for achieving this increased efficiency in flight test programs.

The basic purpose of experimental designs is to enable engineers or scientists conducting experiments or tests to do so in the most efficient manner possible. Experimental designs are constructed to obtain the maximum amount of information about the effects of numerous factors on the output of a process or system in the most economical manner with a minimum possibility of erroneous conclusions. Statisticians have developed a number of standard experimental designs in an attempt to accomplish this basic purpose. Each type of design has particular advantages and limitations associated with it. In general, the simpler designs have wider applicability and require less detailed analysis, but they are not as efficient as some of the more complex designs.

Until recently, the biological, chemical, and agricultural industries have been the only industries that have used experimental designs extensively. However, there is now a trend toward the use of experimental designs in other industries which require extensive testing of their products.

The primary difference between the classical approach and the experimental design approach to conducting tests is the manner of testing the variables. The classical approach allows only one variable at a time to vary while all others are held constant. The experimental design approach permits several or all variables to vary simultaneously. The classical approach is less efficient since less information is gained from the same number of tests. Experimental designs can be used in flight-test programs for two purposes: 1) to reduce the number of tests required to obtain a given amount of data, thereby reducing costs; and 2) to obtain more useful data from a given number of tests in situations where the number of tests cannot be reduced appreciably. Both of these situations are illustrated in this paper.

Investigation has shown that a small number of experimental designs are adequate to cover almost all flight testing requirements. Four general designs have been developed for use in flight testing. Computer programs to process the data from these designs have been established. An ancillary benefit of using experimental designs is the ability to conduct much of the analysis of test data on high-speed digital computers.

Several typical flight-test programs are used to illustrate experimental design techniques and to illustrate the increased efficiency that can be realized through these techniques. The Engineering Flight Test Division of Lockheed-California Company has successfully used these techniques on several programs.

These techniques apply to the testing of missile and space systems, as well as to aircraft testing. The savings realized through the use of these techniques would be even greater for missile and space system testing because of the higher cost of tests and the greater number of factors and interactions that must be considered. The examples presented in this paper

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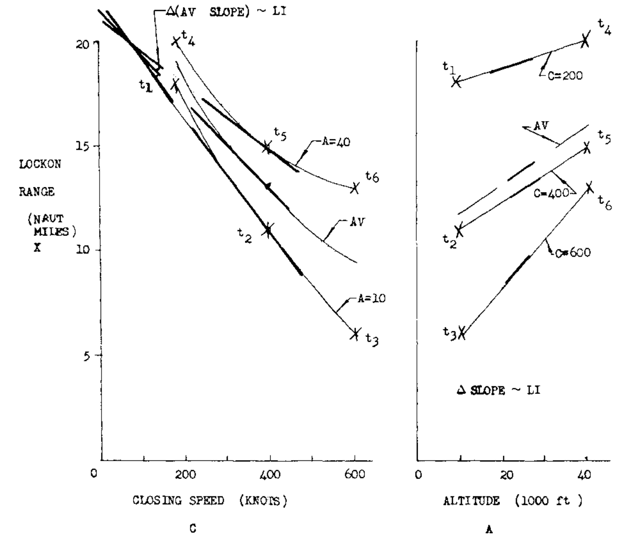

Fig. 1 Effects of altitude and closing speed on lockon range.

are all of aircraft testing only because these are the primary concern of the Engineering Flight Test Division at Lockheed-California Company. The techniques are also equally valid and advantageous for laboratory testing.

The four general designs, which have been developed for use in flight testing, are randomized blocks, partially balanced incomplete blocks, general multiple classifications with interaction, and quadratic analysis of covariance.

Only highlights of the mathematics involved in analyzing data through the use of experimental designs can be presented in this paper. Details can be found in any standard text on the subject.[1,2]

Randomized Block Design

The randomized block design is one of the more basic, general designs. As such, it has extremely broad applicability. The essential characteristic of the randomized block design consists of separating the measured outputs (values of the output for different factor combinations) into blocks. Each block contains one output from each of the factor combinations possible. A factor combination is a particular value for each of the variables affecting the output. Blocks are selected such that all outputs in that block have the greatest possible homogeneity, thereby minimizing experimental error variance. For example, only outputs from one test flight would be put into one block, or, if possible, only outputs from one run of a test flight would be put into one block. By doing this, one can compare the effects of the various factors and estimate them under conditions that are as similar as possible, and the block to block variation may be extracted from the experimental error variance. The only restriction on randomized block designs is that each replication or block must contain one output from each of the possible factor combinations.

The basic mathematical model of a randomized block design is $X_{ij} = \mu + b_i + t_j + e_{ij}$, where X_{ij} is the output, μ is the over-all mean level, b_i is the effect of the ith block, t_j is the effect of the jth treatment (factor combination), and e_{ij} is the experimental error. The following assumptions are made:

$$\sum_i b_i = 0 \qquad \sum_j t_j = 0$$

the e_{ij}'s are uncorrelated and have the same variance σ^2.

The information gained from a randomized block design is as follows: 1) minimum variance, unbiased estimates of the mean output, difference between mean output and any block average output, difference between mean output and average output of any linear combination of factor combinations, and experimental error variance; and 2) significance level of any linear combination of factor combinations, significance level of block to block variation.

A highly simplified example will be used to illustrate the use of a randomized block design in flight test. Consider the problem of determining the effects of altitude and closing speed on the lockon range X_{ij} of an airborne fire control radar. Assume for purposes of this example that the effects of all other factors are negligible and that the effect of altitude is known to be linear. Select two levels of altitude (10,000 and 40,000 ft) and three levels of closing speed (200, 400, and 600 knots). These are 2×3 or 6 possible factor combinations: A_0C_0, A_0C_1, A_0C_2, A_1C_0, A_1C_1, and A_1C_2, where A_0C_0, for example, represents an altitude of 10,000 ft and a closing speed of 200 knots. Each of the six factor combinations is assigned a treatment number 1 through 6 as previously listed (i.e., $t_1 = A_0C_0$, etc.). Assume for purposes of illustration that there is enough fuel to conduct six runs on each test flight, where each run corresponds to one of the factor combinations given. Each flight is considered a block.

There are $6 - 1 = 5$ independent degrees of freedom in the treatment estimates. The sixth degree of freedom is used by the assumption $\sum_j t_j = 0$. This is similar to having six independent equations in six unknowns. The five independent estimates that we desire in this case are: 1) $t_4 + t_5 + t_6 - (t_1 + t_2 + t_3)$, which is the linear effect of altitude; 2) $t_3 + t_6 - (t_1 + t_4)$, which is the linear effect of closing speed; 3) $t_3 + t_6 - 2(t_2 + t_5) + t_1 + t_4$, which is the quadratic effect of closing speed; 4) $t_1 + t_6 - (t_3 + t_4)$, which is the linear interaction between altitude and closing speed; and 5) $t_6 + t_4 - 2t_5 - (t_3 + t_1 - 2t_2)$, which is the quadratic interaction between altitude and closing speed. Assume M flights are made. There will be $M - 1$ degrees of freedom for estimating the block constants and $(M - 1)$ 5 degrees of freedom remaining for estimation of the experimental error variance.

To clarify the meaning of linear and quadratic effects and linear and quadratic interactions, consider the hypothetical results of the foregoing example shown in Figs. 1–3. Six data points have been obtained, one for each of the factor combinations discussed previously. Each of the data points is represented by x's in the figures. Figures 1 and 2 are plots of lockon range vs closing speed with cross plots of lockon range vs altitude. Figure 3 does not have a cross plot since it is not needed to illustrate the quadratic interaction. These figures present the geometric interpretation of each of the effects and

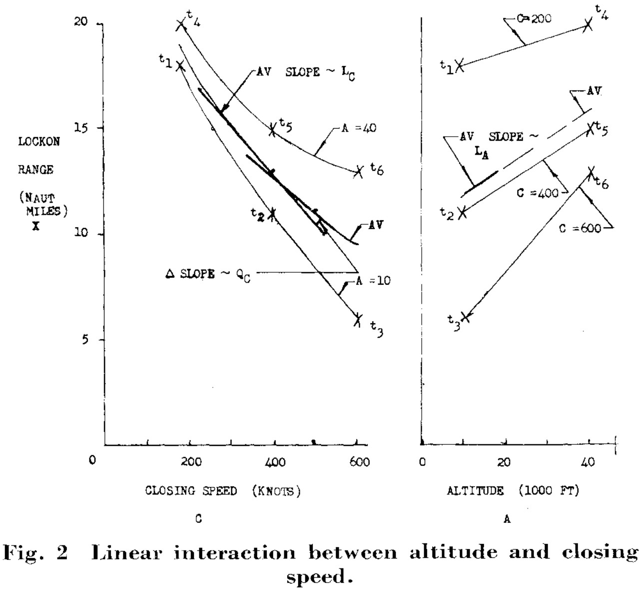

Fig. 2 Linear interaction between altitude and closing speed.

interactions. In addition, if it is desired to determine coefficients of equations such as: lockon range $= U + V$ (closing speed) $+ W$ (closing speed)^2; lockon range $= P + Q$ (altitude), the coefficients U, V, W, P, and Q may be solved in terms of the effects and interactions.

As a result of using the randomized block design in these tests, the following information is available: minimum variance, unbiased estimates of linear effect of altitude, linear and quadratic effects of closing speed, linear and quadratic interactions between altitude and closing speed, flight to flight variation, and experimental error variance. In addition, the significance level of each of the effects and interactions is available. In contrast to this amount of information, consider the information that would be obtained from these tests using present flight-test method of analysis. Plots of lockon range vs closing speed at constant altitude would be made. Cross plots of lockon range vs altitude at constant closing speed also would be made. The only information available from these plots is a hand-faired estimate of the effects of closing speed at fixed altitudes and altitude at fixed closing speeds and a visual measure of the experimental error from the scatter of the data points. Using present methods, the analyst has no accurate measure of the interaction between the two factors, nor any means of accurately separating flight variation from experimental error variance. Furthermore, he has only an intuitive feel for the significance of each effect. This discussion illustrates how much more information can be gained from the same number of tests by using experimental design techniques. From another point of view, consider the case where these tests were conducted without an experimental design in mind. Assume five flights were made, each having two runs at each of the three closing speeds, and each flight was made at a different altitude. Using two flights, one at each extreme of the altitudes used in the five flights, and a randomized block design, one can estimate both effects with the same accuracy as with the five flights which were not planned properly

Engineering flight test at Lockheed-California Company has used the randomized block design computer program to conduct a statistical analysis of radar Contour Mapping Mode errors for the F-104G and F-104G(MAP) airplanes.

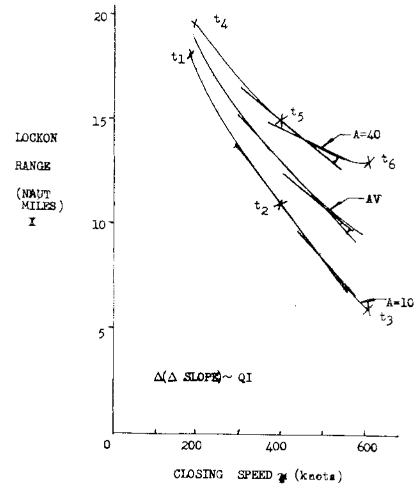

Fig. 3 Quadratic interaction between altitude and closing speed.

Partially Balanced Incomplete Block Design

The partially balanced incomplete block design is a reasonably complicated design, but it has fairly broad applicability. Since it is complicated, the calculations required to analyze this design are fairly complex. Flight test has need for this design because of the one restriction of randomized block designs, which is the necessity for having the block size equal to the total number of possible factor combinations. As discussed previously, a block should be selected so as to provide as much homogeneity as possible, thereby reducing experimental error variance. Therefore, in flight test a block in an experimental design will normally consist of one flight or one run in a flight. If the total number of factor combinations cannot be made equal to the number of data points in one flight or one run, a randomized block design cannot be used to analyze the data. The partially balanced incomplete block design circumvents this difficulty.

The basic feature of the partially balanced incomplete block design is the ability to use block sizes that are smaller than the total number of possible factor combinations. The specifications of this design are as follows: there are t total factor combinations possible; each of these factor combinations is repeated (measured) r times; and they are placed in b blocks each of size K. With reference to any specified factor combination, the remaining $t - 1$ combinations fall into m sets. A member of the ith set ($i = 1, 2, \ldots, m$) occurs with the specified combination in λ_i blocks, and there are n_i factor combinations in the ith set:

$$\left(\sum_{i=1}^{m} n_i = t - 1\right)\left[\sum_{i=1}^{m} \lambda_i n_i = r(K - 1)\right]$$

The numbers n_i are fixed constants no matter which factor combination is specified. There are also relationships between the number of combinations that are common to each of the m sets, but these are too complicated to discuss here. Basically, when the numbers t, r, b, k, m, and λ_i are specified, a specific design is established.

After reviewing the requirements of flight-test programs, it was determined that only designs with one or two sets (i.e., $m = 1$ or $m = 2$) would be required. These sets are called associates. Therefore, equations were established for computer programing of the analysis of data from a partially balanced incomplete block design with two sets of associates. The case of one set of associates is called a balanced incomplete block design. This design can be used as a special case of the two set case. The balanced design usually necessitates a larger number of replications of the factor combinations than is possible in flight test, because of the high cost of tests. The information gained through a partially balanced incomplete design is essentially the same as that listed under randomized block designs.

A distinguishing feature of the partially balanced incomplete block design is the use of interblock, in addition to intrablock, information about the effects of factor combinations. In most designs only intrablock information (comparisons within blocks) is used in estimating the effects of factor combinations. In fact, if only intrablock information is used in a partially balanced design, it is analyzed in exactly the same manner as a randomized block design. Of course, not using the interblock information (comparisons between blocks) makes the design less efficient.

The basic mathematical model of a partially balanced design is $X_{ij} = \mu + b_i + t_j + e_{ij}$ for intrablock, where the symbols and assumptions are the same as for the randomized block design, and

$$B_i = K\mu + Kb_i' + \sum_j \delta_{ij} t_j$$

for interblock, where B_i is the total output of the ith block, K is the size of the block, b_i' is the error of the ith block, $\delta_{ij} = 1$ if treatment j appears in the ith block and 0 if treatment j doesn't appear in the ith block, and t_j is the effect of the jth treatment (factor combination).

The following assumptions are made in the interblock model:

$$\sum_j t_j = 0$$

the b_i''s are uncorrelated with each other and with the e_{ij}'s and have the common variance of $1/K\cdot(\sigma^2 + K\sigma_b^2)$.

Consider a simplified example of the use of a partially balanced design in flight tests. For purposes of illustration and comparison, consider the same example as the one used to illustrate a randomized block design. Now assume that the effect of altitude is not known to be linear. Therefore, we must select three levels of altitude and three levels of closing speed. We will use the same levels as before and add 25,000 ft as the third level for altitude. Now there are nine possible factor combinations: $A_0C_0, A_0C_1, A_0C_2, A_1C_0, A_1C_1, A_1C_2, A_2C_0, A_2C_1$, and A_2C_2. Each of the nine combinations is assigned a treatment number 1 through 9, as listed previously. Now assume that there is enough fuel to conduct only three runs on each test flight, where each run corresponds to a factor combination. Therefore, we want a partially balanced design for nine treatments in blocks of size three. Suppose we allow two classes of associates: $\lambda_1 = 1$, $\lambda_2 = 0$. This means that the factor combinations in the first class of associates will appear in one block together, and the factor combinations in the second class of associates will not appear together in any block. Suppose we take $n_1 = 6$ and $n_2 = 2$ ($n_1 + n_2 = t - 1 = 8$). By the restriction

$$\sum_{i=1}^{m} \lambda_i n_i = r(K - 1)$$

we get $n_1 = 2r$, therefore, $r = 3$. To summarize the design: there are six first associates of each treatment (factor combination) and two second associates; first associates appear in one block together and second associates appear in no blocks together; each factor combination appears three times, and the total number of blocks is nine ($b = rt/K$).

Using the foregoing information in addition to complicated relationships between the number of factor combinations in each class of associates, the following factor combinations appear in each of the nine blocks: (1, 2, 3), (1, 6, 4), (1, 7, 5), (6, 8, 3), (6, 9, 5), (7, 8, 4), (7, 9, 3), (2, 8, 5), and (2, 9, 4). This means the factor combinations shown in Table 1 would be used on each of the nine flights.

Table 1 Factor combinations used on the nine flights

Flight	Run	Altitude, kft	Closing speed, knots
1	1	10	200
1	2	10	400
1	3	10	600
2	1	10	200
2	2	25	600
2	3	25	200
3	1	10	200
3	2	40	200
3	3	25	400
4	1	25	600
4	2	40	400
4	3	10	600
5	1	25	600
5	2	40	600
5	3	25	400
6	1	40	200
6	2	40	400
6	3	25	200
7	1	40	200
7	2	40	600
7	3	10	600
8	1	10	400
8	2	40	400
8	3	25	400
9	1	10	400
9	2	40	600
9	3	25	200

This time there are eight independent degrees of freedom in the treatment estimates. The ones we desire are the five discussed in the example using randomized blocks plus the quadratic effect of altitude and two more interaction effects.

To illustrate the application of experimental design techniques to structural flight tests consider the simplified example of roller coaster tests on an aircraft which is instrumented to measure wing shear, bending and torsion moments, tail loads, fuselage loads, etc. The effects of aircraft configuration (different external stores and external fuel tanks), Mach number, and altitude on the previous loads are desired as a function of load factor. Assume there are six configurations to be tested and three altitudes to be used. Since the effect of Mach number is known not be a simple second order function, five levels of M are used. Because each flight can only be flown in one configuration, all factor combinations cannot be measured in one flight; therefore, a partially balanced incomplete block design would be used instead of a randomized block design.

General Multiple Classifications with Interaction

General multiple classifications with interaction is about the most general experimental design available. Any mathematical model of the form $\mathbf{Y} = X\boldsymbol{\beta} + \mathbf{E}$ can be used, where $\mathbf{Y}$ is an $N \times 1$ vector of observed values (measured quantities), X is an $N \times K$ matrix which specifies the mathematical model used, $\boldsymbol{\beta}$ is a $K \times 1$ vector of coefficients to be estimated, and $\mathbf{E}$ is an $N \times 1$ vector of errors (measurement errors) which are assumed to have zero mean and a common variance σ^2.

The primary information gained from the analysis of general multiple classifications including testing general linear hypotheses is as follows: 1) minimum variance, unbiased estimates of $\boldsymbol{\beta}$, σ^2, variance of $\hat{\boldsymbol{\beta}}$, linear combinations of $\boldsymbol{\beta}$ ($\boldsymbol{\alpha} = L\cdot X\cdot\boldsymbol{\beta}$), variance of $\hat{\boldsymbol{\alpha}}$; 2) significance level of any linear combination of $\boldsymbol{\beta}$; 3) tests of the form $\boldsymbol{\beta} = B\boldsymbol{\gamma} + \mathbf{b}$, where B is a $K \times K'$ matrix, $\boldsymbol{\gamma}$ is a $K' \times 1$ vector ($K' \leq K$), and $\mathbf{b}$ is a $K \times 1$ vector; and 4) significance level of the tests in 3). If, in addition, it is assumed that the E_i's are independent and normally distributed, then the estimates in 1) also will be maximum likelihood estimates.

A number of specific examples will be presented to illustrate the generality of this mathematical model. Consider the standard regression model $y = a + bZ + cZ^2$. This represents one row of the general model by letting $X_{i1} = 1$, $X_{i2} = Z$, $X_{i3} = Z^2$, $i = 1, \ldots, n$, if n values of y are measured. The general model will estimate

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Table 2 Three theoretical test flights, each with three runs

Test	Run	Treatment	Altitude, kft	Closing speed, knots
1	1	t_1	10	200
1	2	t_4	40	200
1	3	t_6	40	600
2	1	t_3	10	600
2	2	t_4	40	200
2	3	t_5	40	400
3	1	t_2	10	400
3	2	t_5	40	400
3	3	t_6	40	600

One can assume y is a linear combination of any function of any number of variables such as $y_i = a_i \cos \theta_i + c_i \cos 2\theta_i + dZ + eZ^2 + fW^{1/2}$ by letting $X_{i1} = 1$, $X_{i2} = \cos\theta_i$, $X_{i3} = \cos 2\theta_i$, $X_{i4} = Z$, $X_{i5} = Z^2$, and $X_{i6} = W^{1/2}$. Then, by use of the general model, a_i's, b_i's, c_i's, d, e, and f are estimated. In addition, hypotheses, such as all a_i's $= 0$, $b_i = b$ for all i, $e = K$, etc., can be tested.

This general model also can be used to analyze factorial designs, such as the example used in the randomized block design illustration. Suppose, for illustration purposes, that because of cost, time, and other considerations, only three test flights were made, each having three runs as shown in Table 2. The specified model of this example (Table 2) is $y_i = \mu + b_i + t_j + e_{ij}$. The general mathematical model for this group of tests, if $\mathbf{Y} = X\boldsymbol{\beta} + \mathbf{E}$, is

$$\begin{bmatrix} y1 \\ y2 \\ y3 \\ y4 \\ y5 \\ y6 \\ y7 \\ y8 \\ y9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \\ \beta_{10} \end{bmatrix} + \begin{bmatrix} E1 \\ E2 \\ E3 \\ E4 \\ E5 \\ E6 \\ E7 \\ E8 \\ E9 \end{bmatrix}$$

The computer program would then estimate β_1 through β_{10}; $\beta_1 = \mu$, $\beta_2 = b_1$, $\beta_3 = b_2$, $\beta_4 = b_3$, $\beta_5 = t_1$, $\beta_6 = t_2$, $\beta_7 = t_3$ $\beta_8 = t_4$, $\beta_9 = t_5$, and $\beta_{10} = t_6$.

The program also would estimate any linear combination of the treatment effects. In this case the same combinations as listed for this example under the randomized block design section would be estimated. Thus, we see that the same estimates could be made without using a randomized block design; however, the estimates would be considerably less accurate since all of the effects of factor combinations were not measured under uniform conditions (on one flight), and since only three flights were made there are fewer degrees of freedom for estimating the experimental error.

To illustrate the testing of hypotheses of the form $\boldsymbol{\beta} = B\boldsymbol{\gamma} + \mathbf{b}$, consider the hypothesis $\beta_2 = \beta_3 = \beta_4 = 0$ (i.e., there is no flight to flight variation). Then, $\boldsymbol{\beta} = B\boldsymbol{\gamma} + \mathbf{b}$ becomes

$$\begin{array}{cccccc} \boldsymbol{\beta} & & B & & \boldsymbol{\gamma} & & \mathbf{b} \\ \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \\ \beta_{10} \end{bmatrix} & & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} & \cdot & \begin{bmatrix} \beta_1 \\ \beta_1 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \\ \beta_{10} \end{bmatrix} & + & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$

As an example where $\mathbf{b}$ would not be zero, suppose you want to find out if the mean lockon range over all altitudes and closing speeds is 10 naut miles. Then $\boldsymbol{\beta} = B\boldsymbol{\gamma} + \mathbf{b}$ would become

$$\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \\ \beta_{10} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \\ \beta_6 \\ \beta_7 \\ \beta_8 \\ \beta_9 \\ \beta_{10} \end{bmatrix} + \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Quadratic Analysis of Covariance

Basically, the use of experimental designs requires that the factors affecting a measured output be at fixed levels during a test. Many times there are factors affecting a measured output which cannot be directly controlled (e.g., wind direction and speed, ambient temperature, etc.). The effects of these uncontrolled factors can be determined by the analysis of covariance. Because this situation arises frequently, a special computer program that combines a randomized block design with quadratic analysis of covariance has been established. The general multiple classifications with interaction program can be used for general analysis of covariance problems that cannot be fitted to a randomized block design.

The basic mathematical model of the quadratic analysis of covariance program is: $y_{ij} = \mu + b_i + t_j + \alpha(X_{ij} - \bar{X}) + B(X_{ij}^2 - \bar{X}^2) + e_{ij}$, where y_{ij} is the measured output, X_{ij} is the value of the uncontrolled factor, $\bar{X}$ is the average value of X_{ij} over all i and j, μ is the over-all mean level of y_{ij}, b_i is the effect of the ith block, t_j is the effect of the jth treatment (controlled factor combination), α is the linear covariance coefficient, B is the quadratic covariance coefficient, and e_{ij} is the experimental error. The following assumptions are made: $\sum_i b_i = 0$ and $\sum_j t_j = 0$, and the e_{ij}'s are uncorrelated and have the same variance σ^2.

The primary information obtained from the quadratic analysis of covariance program is: 1) minimum variance, unbiased estimates of μ, α, B, experimental error variance $t_j - t_k$, and the variance of $t_j - t_k$ and 2) significance level of $t_j - t_j$.

As a simplified example of the use of an analysis of covariance design in flight test, consider longitudinal control tests where you want to measure the effects of altitude, trim speed, speed, external stores, and gross weight on elevator angle. Since gross weight varies continuously and you cannot establish a fixed level of this parameter, it is preferable to use analysis of covariance for gross weight.

Using three altitudes, two trim speeds, three speeds, and with and without external stores, the following information would be obtained from the experimental design: minimum variance, unbiased estimates of linear and quadratic effects of altitude, linear effect of trim speed, linear and quadratic effects of speed, linear effect of external stores, linear and quadratic covariance coefficients of gross weight, interactions between all of the foregoing factors, and experimental error variance. In addition, the significance level of each of the forementioned effects and interaction is available.

To illustrate the savings in number of tests required using an experimental design, suppose that you want to measure each of the foregoing effects n times. Using conventional single factor methods, this would require $10n$ test points; however, using an experimental design, this would require only $6n$ test points. Therefore, you have saved 40% on the number of test points in this particular series of tests.

The quadratic analysis of covariance computer program has been used in flight test at Lockheed to analyze the normal acceleration characteristics of the LMV-8 hydroskiboat model. The primary objective of the analysis was to determine if there was a significant reduction in normal acceleration with skis extended.

To illustrate the application of experimental design techniques to an aircraft performance test program, consider a series of tests to determine the rate of fuel consumption for various operating conditions. Fuel flow is a function of power setting, Mach number, altitude, and ambient temperature. The effects of all of these parameters are known to be nonlinear; therefore, three levels of each factor are selected. Since ambient temperature cannot be directly controlled, analysis of covariance would be used to obtain the effect of this parameter. There are $3^3 = 27$ factor combinations to be measured. Because of insufficient fuel capacity, all 27 combinations cannot be measured on one flight; therefore, a

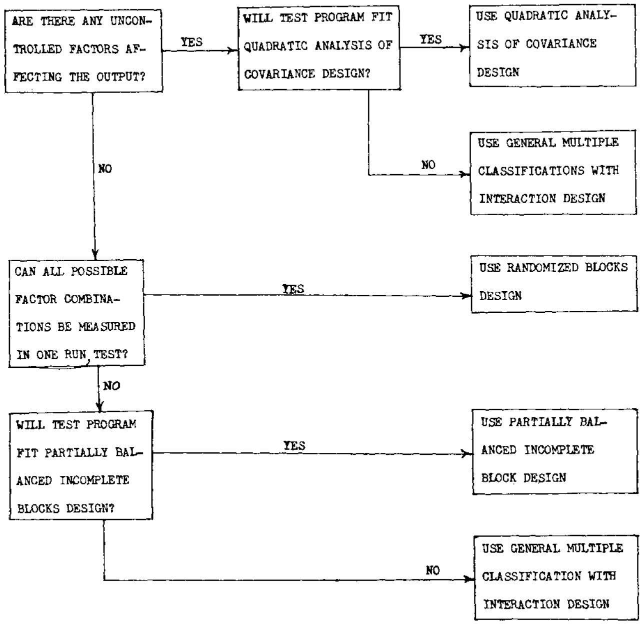

Fig. 4 Decision flow chart.

partially balanced incomplete block design would have to be used in conjunction with analysis of covariance. As previously mentioned, this design could be run on the general multiple classifications with interaction computer program.

Comparison of Designs and Conventional Methods

The question arises as to how you decide which of the four types of designs presented here to use in a particular test program. There are many facets to this question; however, in general, you would use the flow chart in Fig. 4 to decide which design to use.

It is very difficult to present a general rule giving the savings in number of test points that can be achieved by using experimental design techniques, since this depends on how the tests would be conducted using conventional methods and on the particular design used. However, assuming all factor effects are independent, the following rule of thumb can be used. If you want to measure each effect n times, $n \cdot (2x + 3y)$ test points would be required using conventional methods of varying one factor at a time as compared to $6n$ test points required using an experimental design, where x is the number of factors for which two levels are measured and y is the number of factors for which three levels are measured. One can see from the foregoing that, in general, the savings increase as the number of factors affecting an output increase.

Summary

Experimental designs can be used in flight-test programs for two purposes: 1) to reduce the number of tests required to obtain a given amount of data, thereby reducing costs and 2) to obtain more useful data from a given number of tests in situations where the number of tests cannot be appreciably reduced. Four general types of designs have been found to be adequate to accomplish these purposes.

An investigation of specific types of flight-test programs indicate that experimental design techniques are particularly valuable for system performance tests in which it is desired to measure and analyze the effects of several variables on system performance. Both of the forementioned purposes can be readily accomplished in this type of test.

It is advantageous to use experimental design techniques in aerodynamic performance and stability and control tests. However, because of the many restrictions (both physical and government demonstration requirements) on the test levels of the variables, the first purpose of experimental design can be accomplished in only a limited manner. Nevertheless, the second purpose can be accomplished in almost all test programs of this type. Although a thorough investigation of structural test programs has not been completed, preliminary results of this investigation indicate that it is feasible and advantageous to use experimental design techniques in order to accomplish both purposes.

References

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